The Maximum Principle (Hamiltonian)

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Towa Tachibana

Derive the Hamiltonian

We can solve dynamic optimization problems by the Lagrange method.[1] But in economics, we often use an easier and a more intuitive method to tackle dynamic optimizations: the Maximum Principle. There are no essential differences between the Lagrange method and the Maximum Principle. In fact, we can consider the Maximum Principle as a formulation of the Lagrange solutions. We here study how we can derive the Maximum Principle from the Lagrangian. The following explanation is mainly based on Dixit (1990, Ch.10). To brush up, let us rewrite the Command optimum of the Ramsey model.

$$\max \sum_{t=0}^{T} \beta^t u(c_t) \tag{1}$$

$$\text{s.t.} \quad k_{t+1} = (1-\delta)k_t + f(k_t) - c_t, \quad (t = 0, \ldots, T) \tag{2}$$

$$k_0 = \overline{k_0} \qquad \text{(given)} \tag{3}$$

$$k_{T+1} \geq \overline{k_{T+1}} \qquad (>0) \tag{4}$$

[1] Chow (1997) is a strong advocate of the Lagrange method for dynamic problems.

The Lagrangian function for this problem is:

$$
\begin{aligned}
\mathcal{L} = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots + \beta^t u(c_t) + \beta^{t+1} u(c_{t+1}) + \cdots \beta^{T-1} u(c_{T-1}) + \beta^T u(c_T) \\
& + \lambda_1[f(k_0) + (1-\delta)k_0 - c_0 - k_1] \\
& + \lambda_2[f(k_1) + (1-\delta)k_1 - c_1 - k_2] \\
& + \lambda_3[f(k_2) + (1-\delta)k_2 - c_2 - k_3] \\
& \vdots \qquad\qquad\qquad \vdots \\
& + \lambda_{t+1}[f(k_t) + (1-\delta)k_t - c_t - k_{t+1}] \\
& + \lambda_{t+2}[f(k_{t+1}) + (1-\delta)k_{t+1} - c_{t+1} - k_{t+2}] \\
& \vdots \qquad\qquad\qquad \vdots \\
& + \lambda_{T+1}[f(k_T) + (1-\delta)k_T - c_T - k_{T+1}] \\
& + \mu[k_{T+1} - \overline{k_{T+1}}]
\end{aligned}
$$

To derive the Maximum Principle, all we need to do is to re-arrange this Lagrangian.

First we can rearrange it as follows:

$$
\begin{aligned}
\mathcal{L} = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots + \beta^t u(c_t) + \beta^{t+1} u(c_{t+1}) + \cdots \beta^{T-1} u(c_{T-1}) + \beta^T u(c_T) \\
&+ \lambda_1[f(k_0) - \delta k_0 - c_0] + \lambda_1 k_0 - \lambda_1 k_1 \\
&+ \lambda_2[f(k_1) - \delta k_1 - c_1] + \lambda_2 k_1 - \lambda_2 k_2 \\
&+ \lambda_3[f(k_2) - \delta k_2 - c_2] + \lambda_3 k_2 - \lambda_3 k_3 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
&+ \lambda_{t+1}[f(k_t) - \delta k_t - c_t] + \lambda_{t+1} k_t - \lambda_{t+1} k_{t+1} \\
&+ \lambda_{t+2}[f(k_{t+1}) - \delta k_{t+1} - c_{t+1}] + \lambda_{t+2} k_{t+1} - \lambda_{t+2} k_{t+2} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
&+ \lambda_T[f(k_{T-1}) - \delta k_{T-1} - c_{T-1}] + \lambda_T k_{T-1} - \lambda_T k_T \\
&+ \lambda_{T+1}[f(k_T) - \delta k_T - c_T] + \lambda_{T+1} k_T - \lambda_{T+1} k_{T+1} \\
&+ \mu[k_{T+1} - \overline{k_{T+1}}]
\end{aligned}
$$

Note that **the expressions in the brackets indicate** $k_{t+1} - k_t$, which is investment at period t.

We need one more step. The Lagrangian above can be further arranged as follows.

$$\begin{aligned}
\mathcal{L} = {} & \lambda_1 k_0 + u(c_0) + \lambda_1 [f(k_0) - \delta k_0 - c_0] \\
& + [\lambda_2 - \lambda_1] k_1 + \beta u(c_1) + \lambda_2 [f(k_1) - \delta k_1 - c_1] \\
& + [\lambda_3 - \lambda_2] k_2 + \beta^2 u(c_2) + \lambda_3 [f(k_2) - \delta k_2 - c_2] \\
& \vdots \qquad\qquad\qquad\qquad \vdots \\
& + [\lambda_{t+1} - \lambda_t] k_t + \beta^t u(c_t) + \lambda_{t+1} [f(k_t) - \delta k_t - c_t] \\
& + [\lambda_{t+2} - \lambda_{t+1}] k_{t+1} + \beta^{t+1} u(c_{t+1}) + \lambda_{t+2} [f(k_{t+1}) - \delta k_{t+1} - c_{t+1}] \\
& \vdots \qquad\qquad\qquad\qquad \vdots \\
& + [\lambda_{T+1} - \lambda_T] k_T + \beta^T u(c_T) + \lambda_{T+1} [f(k_T) - \delta k_T - C_T] \\
& + [\mu - \lambda_{T+1}] k_{T+1} - \mu \overline{k_{T+1}}
\end{aligned}$$

Finally, we define the Hamilton function (Hamiltonian) as follows. More precisely, we here define the present-value Hamiltonian.

$$H_t = \beta^t u(c_t) + \lambda_{t+1} [f(k_t) - \delta k_t - c_t] \tag{5}$$

With the Hamiltonian, the original Lagrangian becomes very simple:

$$
\begin{aligned}
\mathcal{L} = \lambda_1 k_0 &+ H(c_0, k_0, \lambda_1) \\
&+ [\lambda_2 - \lambda_1]k_1 + H(c_1, k_1, \lambda_2) \\
&+ [\lambda_3 - \lambda_2]k_2 + H(c_2, k_2, \lambda_3) \\
&\vdots \qquad\qquad \vdots \\
&+ [\lambda_{t+1} - \lambda_t]k_t + H(c_t, k_t, \lambda_{t+1}) \\
&+ [\lambda_{t+2} - \lambda_{t+1}]k_{t+1} + H(c_{t+1}, k_{t+1}, \lambda_{t+2}) \\
&\vdots \qquad\qquad \vdots \\
&+ [\lambda_{T+1} - \lambda_T]k_T + H(c_T, k_T, \lambda_{T+1}) \\
&+ [\mu - \lambda_{T+1}]k_{T+1} - \mu\overline{k_{T+1}}
\end{aligned}
$$

References

Chow, Gregory C. 1997. *Dynamic Economics*. New York: Oxford University Press.

Dixit, Avinash K. 1990. *Optimization in Economic Theory*. Oxford: Oxford University Press, 2 ed.