Infinite Horizon, Autonomous, and Steady State

Natural Resource Economics

Fall 2015, Towa Tachibana

First draft: November 11, 2004

Last Revision: December 15, 2015

In the class, we have mainly studied "discrete-time, finite-horizon" problems. In the general form,

$$\max \sum_{t=0}^{T} F(y_t,\ x_t,\ t) \tag{1}$$

$$\text{s.t.} \qquad y_{t+1} - y_t = Q(y_t,\ x_t,\ t) \tag{2}$$

$$y_0 = \overline{y_0}$$

$$y_{T+1} \geq \overline{y_{T+1}}$$

In economics, however, we often consider "continuous-time, infinite-horizon, and autonomous" problems such as

$$\max \int_{t=0}^{\infty} e^{-rt} F(y(t),\ x(t))dt \tag{3}$$

$$\text{s.t.} \qquad \dot{y}(t) = Q(y(t),\ x(t)) \tag{4}$$

$$y(0) = \overline{y(0)}$$

Among these three concepts, we have already studied "continuous-time". In this note, we will examine "infinite-horizon" and "autonomous". In short, infinite-horizon means that the last period of a maximization problem is ∞. Autonomous means that both the objective function and the transition equation do not explicitly depend on time index t. Time enters only through a discounting factor: e^{-rt}. Let's start from "infinite-horizon".

Infinite Horizon

In infinite-horizon problems, the last period of maximization problem is far far far far far $\cdots$ away. Namely, the last period T in equation (1) is replaced by ∞ in equation (3). In infinite-horizon problems, an agent (individual, firm, government) tries to maximize her objective function for a time interval $[0, \infty]$. At first glance, this formulation seems to be stupid (at least, for me)! No individual lives forever (although my first name "Towa" means forever)! Why should we often consider infinite-horizon problems? The reasons or rationales are as follows.

(I) Technically, infinite-horizon problems, in particular infinite-horizon autonomous problems, are easy to analyze.

(II) No individuals live forever. It is true. But, firms and nations are usually "going concern". Firms and nations *assume* that they exist for ever. Explicitly considering the last period T in the firms' or national economies' problems are something like assuming the time of bankrupt or revolution. Furthermore, setting up T means that these firms or nations know the year of bankrupt or revolution. This is a rather strange assumption. In these cases, it is appropriate to assumes infinite horizon.

(III) The argument in (II) does not apply to dynamic optimization problems of individuals. To repeat, no individuals live forever. If individuals are concerned about the welfare of their kids, however, we can assume a person with infinite horizon: a person who lives forever. This is a framework set up by Barro (1974), and is often referred to as the dynasty model. Assume that the utility function of the first generation is defined by

$$U_0 = u(c_0) + \beta U_1$$

Here c_0 is the consumption of the first generation, and U_1 is the welfare of the second generation: children of the first generation. Thus individuals here are altruistic. In addition to their own welfare, they care about the welfare of their children. Similarly, the utility function of the second generation is defined by

$$U_1 = u(c_1) + \beta U_2$$

The third generation:

$$U_2 = u(c_2) + \beta U_3$$

and this process continues. Let us substitute U_1 into U_0. We have

$$\begin{aligned} U_0 &= u(c_0) + \beta[u(c_1) + \beta U_2] \\ &= u(c_0) + \beta\, u(c_1) + \beta^2 U_2 \end{aligned}$$

If we continue, we will find out a kind of individual with infinite horizon.

$$u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots + \beta^t u(c_t) + \beta^{t+1} u(c_{t+1}) + \cdots$$

Autonomous

To your surprise, I guess, that it is more difficult to find out a clear rationale for the assumption of "autonomous" than for infinite horizon. As stated above, "autonomous" means that both the objective function and the transition equation do not explicitly depend on time index t. Time enters only through a discounting factor: e^{-rt}. You should check your notebook now. Both Kobe-city model and the Ramsey model are in the form of autonomous problem.

Let us think about the utility function in the Ramsey model. It is autonomous because we assume $u(c_t)$ not $u(c_t, t)$. That is, the shape of utility function, namely preference, does not change over time. For concreteness, consider an non-autonomous Cobb-Douglas utility function over two goods: fried foods (F) and boiled foods (B). If it describes my preference, it is likely to be:

$$u(F_t, B_t, t) = {F_t}^{\alpha(t)} {B_t}^{1-\alpha(t)}$$

Here t represents my age. In my case, $\alpha(15)$ is about 0.9, but $\alpha(51)$ becomes much smaller, say 0.55. It is natural that people prefer oily food when they are young, but prefer plain food when they become older. In fact, to endogenize the formation of preferences is one of the major research topics in current microeconomics. Autonomous assumption,

$$u(F_t, B_t) = {F_t}^{\alpha} {B_t}^{1-\alpha},$$

discards such changes.

In the cases of production, autonomous assumption means that the shape of production function does not change over time. Note that it does not exclude changes in production technology. For example, we often assume

$$F[K(t), A(t)L(t)]$$

$$A(t)F[K(t), L(t)]$$

where K(t) is capital input, L(t) is labor input, and A(t) is technology level. The first one describes Harrod-neutral technology, and the second one describes Hicks-neutral technology. Although these production functions are autonomous, they include technological changes as the changes in $A(t)$.

But with the autonomous assumption, technological changes occur in the same fashion over time. In reality, we may experience Hicks-neutral technological progress in 2002, while we may observe Harrod-neutral technological progress in 2003. With autonomous assumption, we restrict the model to one of these technological changes.

"Autonomous" is a simplifying assumption. In the long-run problems of a society, which is typical in environment economics, it is *not a so inappropriate* assumption because the average preferences over the society will not change so drastically.

Steady State

In infinite-horizon autonomous problems, there are usually a terminal situation where variables in the problem settle down to growing at constant rates. Such situation is referred to as a "steady state". The system moves toward to a steady state to fulfill the objective of the dynamic optimization problem. In figure 2.4 of the Romer (2001, p.58), for example, the steady state is point E. Note that a steady state is a point on an equilibrium path. In other words, a steady state is one of the equilibrium. In figure 2.4 of Romer (2001, p.58), equilibrium of the system is the path FE, not only the point E.

Example

Let us solve an infinite-horizon autonomous problem. It is a version of Kobe-city problem. For notations, refer to your note and Nalebuff (1997).

$$\max \int_{t=0}^{\infty} e^{-rt}\{u(P(t)) + F(p(t))\} \tag{5}$$

$$\text{s.t.} \quad \dot{P}(t) = p(t) - \delta P(t), \tag{6}$$

$$P(0) = \overline{P(0)}$$

In this example, I use the interest rate r as the discount factor. Hereafter, we drop t for simpler exposition. The current value Hamiltonian of this model is:

$$\mathcal{H} = u(P) + F(p) + m[p - \delta P]$$

The FOCs are

$$\mathcal{H}_p = F'(p) + m = 0 \tag{7}$$

$$\dot{m} = rm - \mathcal{H}_P = rm - [u'(P) - \delta m] \tag{8}$$

$$\dot{P} = p - \delta P \tag{9}$$

$$\lim_{t\to\infty} mPe^{-rt} = 0 \tag{10}$$

Equation (10) is the transversality condition of this infinite-horizon problem. In this example, it meams that the discounted present value of polluted sediment in infinitely far-away future, which is usually negative, should be zero. This interpretation may not be intuitive. In the infinite-horizon Ramsey model, you can find out an intuitive interpretation of the transversality condition.

From equation (7), we can calculate

$$\dot{m} = -F''(p)\dot{p} \tag{11}$$

Substitute this into equation (8). We have

$$-F''(p)\dot{p} = -F'(p)(r + \delta) - u'(P) \tag{12}$$

Equations (12) and (9) consist of Hamiltonian dynamics, in which we can consider the steady state. Let us consider the steady state where $\dot{p} = \dot{P} = 0$. By substituting $\dot{p} = \dot{P} = 0$ into equations (12) and (9), we obtain

$$F'(\delta P_s) + \frac{u'(P_s)}{r + \delta} = 0,$$

where P_s is the steady-state level of polluted sediment P. This condition tells that at the social optimum, the benefit from production (more pollution emmision) should be equal to the discounted welfare loss from polluted sediment.

Phase Diagram

One more important topic in dynamic optimization problems is how to draw phase diagrams. I do not have time to explain it in the class. Please refer to the explanatin in Romer (2001, Ch.2).

References

Barro, Robert J., "Are Government Debts Net Worth?," *Journal of Political Economy*, 1974, *81* (6), 1095–1117.

Nalebuff, Barry, "On a Clear Day, You Can See the Coase Theorem," in Partha Dasgupta and Karl-Göran Mäler, eds., *The Environment and Emerging Development Issues Vol. I*, Oxford: Clarendon Press, 1997, pp. 35–47.

Romer, David, *Advanced Macroeconomics*, 2 ed., New York: McGraw-Hill, 2001.